Nonlinear Elimination Preconditioned Newton's Method for Hyperelasticity with Application in Arterial Wall simulations

Shihua Gong[1] and Xiao-Chuan Cai[2]

1. Peking University, Beijing, China; 2. University of Colorado, Boulder, CO, USA

Anisotropic Hyperelastic Model

Arterial wall can be modeled by a nearly incompressible, anisotropic and hyperelastic equation that allows large deformation.

- Energy Functional

$$\psi = \psi^{iso}(C) + \psi^{vol}(C) + \psi^{ti}(C, M^{(i)}), \quad (1)$$

where C is Cauthy-Green tensor, $M^{(i)}$ are the structural tensors.

- Principal Invariants

$$I_1 := \text{tr } C, \quad I_2 := \text{tr } [\text{cof} C], \quad I_3 := \det\ C,$$
$$J_4^{(i)} := \text{tr}[CM^{(i)}], \quad J_5^{(i)} := \text{tr}[C^2 M^{(i)}].$$

with $\psi^{iso} = \psi^{iso}(I_i)$, $\psi^{vol} = \psi^{vol}(I_3)$ and $\psi^{ti} = \psi^{ti}(I_i, J_j^{(i)})$.

- Momentum Equation

$$\text{div} P = -f, \quad (2)$$

where $P = FS$, $S = \frac{\partial \psi}{\partial C}$.

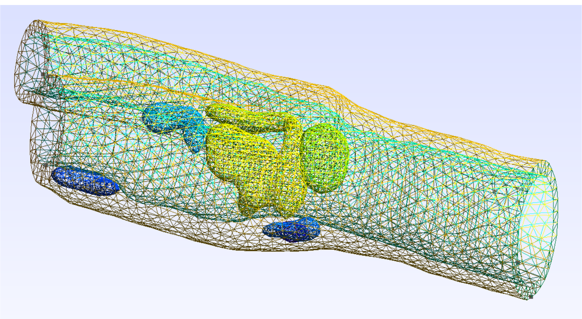


Figure 1: A carotid artery with plaques

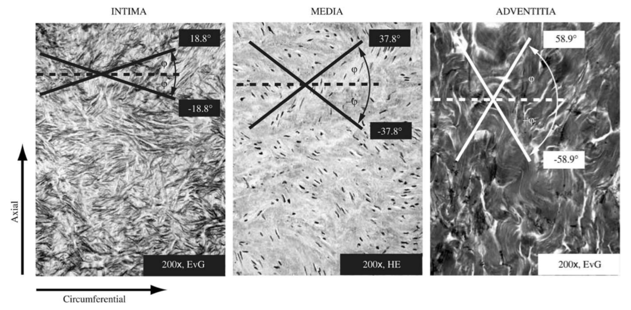


Figure 2: Collagen fibre reinforcement

To solve nonlinear system (2), the performance of Inexact Newton methods (IN) and the linear solvers degrade in the cases of

Large Deformation; Near Incompressibility; High Anisotropy.

We use an overlapping Schwarz preconditioner and propose a nonlinearly preconditioned Newton's method based on nonlinear elimination to accelerate the convergence of linear and nonlinear iterations, respectively.

Nonlinear Preconditioning Based on Nonlinear Elimination

Denote the nonlinear system discretized from (2) by

$$F(u^*) = 0$$

where $F : \mathbf{R}^n \mapsto \mathbf{R}^n$. Newton's method finds a sequence of improving approximate solutions iteratively $u^{(k+1)} = u^{(k)} - \left(F'(u^{(k)})\right)^{-1} F(u^{(k)})$.

- Convergence of Newton's method:

$$e^{(k+1)} = -\left(F'(u^{(k)})\right)^{-1} \left\langle \frac{1}{2} F''(u^{(k)}) e^{(k)}, e^{(k)} \right\rangle + \mathcal{O}(\|e^{(k)}\|^3).$$

Here $e^{(k)} = u^* - u^{(k)}$ is the error of the kth approximate solution.

- Key idea:
  Eliminate some "subfunctions" of F to balance the overall nonlinearity.
- Quantitative characterization of the nonlinearity:

$$F(u^{(k+1)}) = F(u^{(k)}) + F'(u^{(k)}) p^{(k)} + \left\langle \frac{1}{2} F''(u^{(k)} + \theta p^{(k)}) p^{(k)}, p^{(k)} \right\rangle$$
$$\approx \left\langle F''(u^{(k)} + \theta p^{(k)}) p^{(k)}, p^{(k)} \right\rangle.$$

High nonlinearity $\sim$ Large residual.

- Nonlinear elimination:
  - Find "bad" DOF set S_b from $S = \{1, \cdots, n\}$, according to the residual

$$V_b = \{v \mid v = (v_1, \cdots, v_n)^T \in \mathbf{R}^n, v_k = 0, \text{ if } k \notin S_b\}.$$

  - Given an approximation u, NE finds correction by solving $u_b \in V_b$ such that

$$F_b(u_b) := R_b F(u_b + u) = 0.$$

Inexact Newton with Nonlinear Elimination Preconditioner

Algorithm (IN-NE)

Step 1. Compute the next approximate solution $u^{(k+1)}$ by solving

$$F(u) = 0$$

with one step of IN iteration using $u^{(k)}$ as the initial guess.

Step 2. (Nonlinearity checking)

2.1 If $\|F(u^{(k+1)})\| < \varrho_1 \|F(u^{(k)})\|$, go to Step 1.

2.2 Finding "bad" d.o.f. by

$$S_b := \{j \in S \mid |F_j(u^{(k+1)})| > \varrho_2 \|F(u^{(k+1)})\|_\infty\}.$$

And extend S_b to S_b^δ by adding the neighboring DOFs.

2.3 If $\#(S_b^\delta) < \varrho_3 n$, go to Step 3. Otherwise, go to Step 1.

Step 3. Compute the correction $u_b^\delta \in V_b$ by solving the subproblem approximately

$$F_b^\delta(u_b^\delta) := R_b^\delta F(u_b^\delta + u^{(k+1)}) = 0,$$

with an initial guess $u_b^\delta = 0$.Update $u^{(k+1)} \leftarrow u_b^\delta + u^{(k+1)}$. Go to Step 1.

- Three control parameters:
  - ϱ_1 the tolerance for the reduction of the residual norm;
  - ϱ_2 the tolerance to pick up the bad variables and equations;
  - ϱ_3 the tolerance to limit the size of the subproblem.
- Boundary effect:
  - If the nonlinear elimination just on S_b, the residual near the boundaries of the eliminating domains would become very large.
  - To ease this phenomenon, we extend the index set S_b to S_b^δ by adding the neighboring DOFs, of which the distances to S_b are smaller than δ.

Test Examples

We consider the polyconvex energy functional

$$\psi_A = \psi^{isochoric} + \psi^{volumetric} + \psi^{ti}$$
$$:= c_1 \left(\frac{I_1}{I_3^{1/3}} - 3 \right) + \epsilon_1 \left(I_3^{\epsilon_2} + \frac{1}{I_3^{\epsilon_2}} - 2 \right) + \sum_{i=1}^{2} \alpha_1 \left\langle I_1 J_4^{(i)} - J_5^{(i)} - 2 \right\rangle^{\alpha_2}.$$

Based on the parameter sets of the model ψ_A in Table. 1, we propose three test examples to investigate the performance of our algorithms for the case of large deformation, near incompressibility and high anisotropy.

Set	Layer	c_1(kPa)	ϵ_1(kPa)	ϵ_2(-)	α_1(kPa)	α_2	Purpose
L	–	1.0	1.0	1.0	0.0	0.0	Deformations by different pulls
C1	–	17.5	4.998	2.4	0.0	0.0	
C2	–	17.5	49.98	2.4	0.0	0.0	Different penalties for compressiblity
C3	–	17.5	499.8	2.4	0.0	0.0	
A1	Adv.	7.5	100.0	20.0	1.5e10	20.0	
	Med.	17.5	100.0	50.0	5.0e5	7.0	
A2	Adv.	6.6	23.9	10	1503.0	6.3	Anisotropic arterial walls
	Med.	17.5	499.8	2.4	30001.9	5.1	
A3	Adv.	7.8	70.0	8.5	1503.0	6.3	
	Med.	9.2	360.0	9.0	30001.9	5.1	

Table 1: Model parameter sets of ψ_A.

The first example simulates the deformations of a cylindrical rod by different pulls $L_1 = 1.\text{e}1$ Pa, $L_2 = 1.\text{e}2$ Pa and $L_3 = 1.\text{e}3$ Pa. The rest examples simulate the artery walls imposing blood pressure 12 kPa.

Numerical Results

- The simulation results for the three examples are depicted as follows

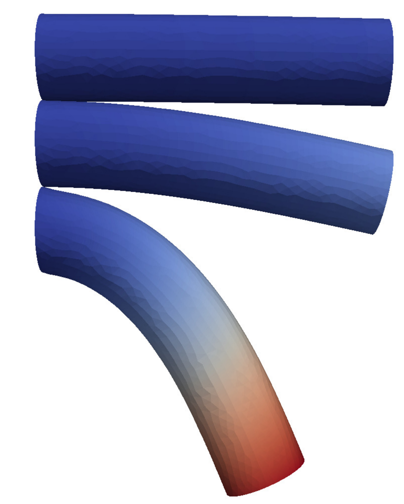


Figure 3: Deformations by different pulls

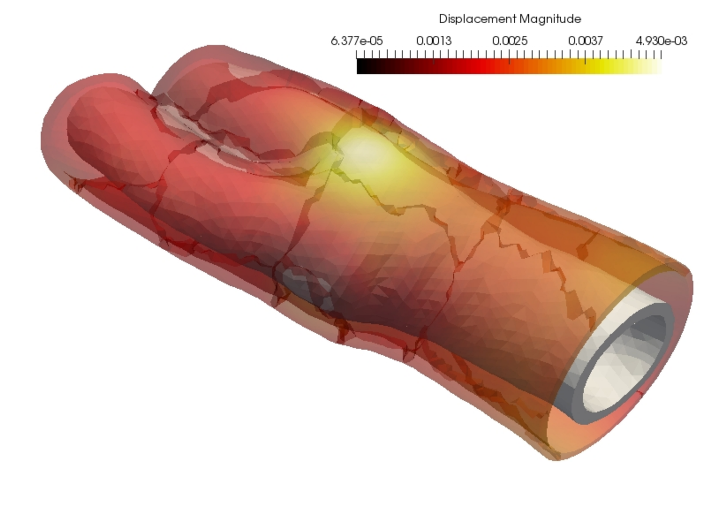


Figure 4: A diastolic carotid artery

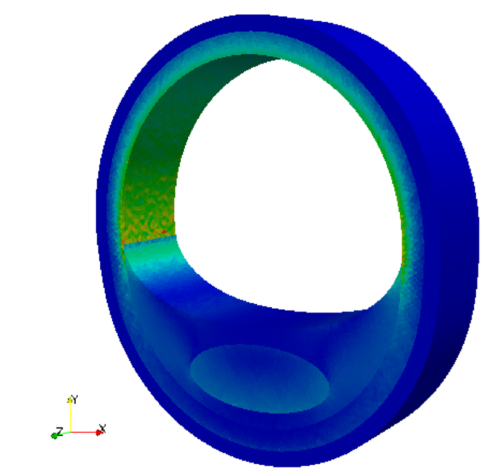
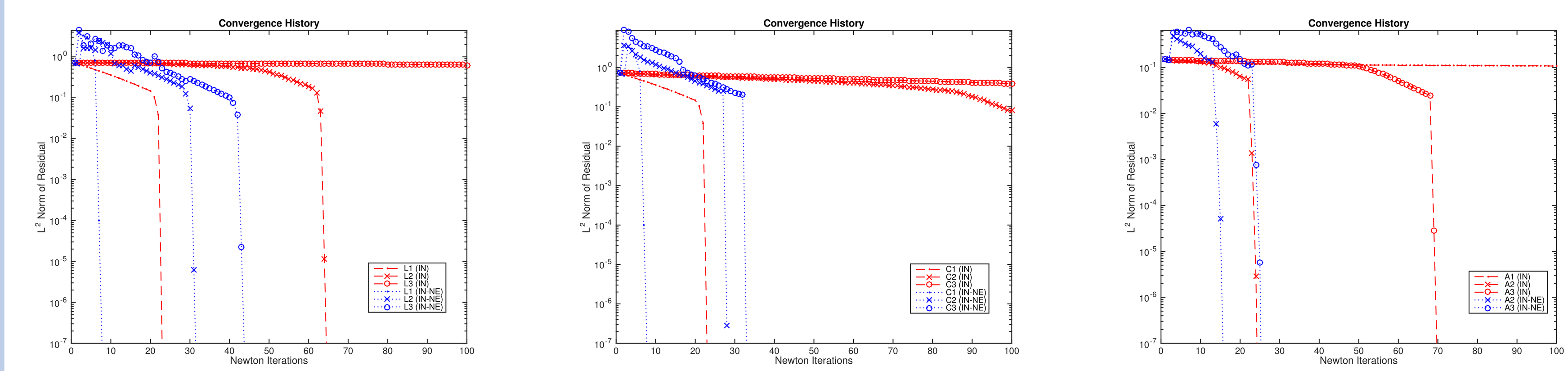


Figure 5: A fibre-reinforced multilayer artery

- The convergence history



(a) Example L (b) Example C (c) Example A

Figure 6: Convergence history of IN and IN-NE

- The parameter-sensitivity tests

$\varrho_2 = .9, \varrho_3 = .3$			
ϱ_1	.9	.95	.98
Global Newton iterations	23	23	24
Total Newton iterations of NE	40	25	23
$\rho_1 = .95, \rho_3 = .3$			
ϱ_2	.8	.9	.95
Global Newton iterations	24	23	25
Total Newton iterations of NE	31	25	23
$\rho_1 = .95, \rho_2 = .9$			
ϱ_3	.1	.2	.3
Global Newton iterations	54	23	23
Total Newton iterations of NE	15	25	25

Table 2: Number of iterations of IN-NE with respect to different pre-chosen parameters.

Set	Poisson's Ratio
C1	0.125
C2	0.452
C3	0.495

Table 3: Poisson's ratio of materials C1, C2 and C3.

mesh \ δ	2	3	4
m_0	15	23	23
m_1	58	36	26

Table 4: Mesh refinement, Set A2, $\rho_0 = .9$, $\rho_1 = .9$, $\rho_2 = .25$.

Concluding Remarks

- We investigated the performance of a nonlinear elimination preconditioner with applications in computational hyperelasticity.
- A robust strategy of nonlinearity checking was adapted to capture the subregions with stronger nonlinearity, which coincide with the propagation of the elastic wave.
- We found that the extension for the eliminating index set by adding the neighboring DOFs is an effective trick to ease the thrashing phenomenon of nonlinear elimination.