Optimal Multilevel Solvers and New Hybridized Mixed Methods for Linear Elasticity

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Abstract
We present a family of new mixed finite element methods for the linear elasticity and then develop multilevel solvers to solve the linear system. Our mixed discretization, preserving the symmetry and the $H(div)$ conformity in the stress approximation, can be efficiently implemented by hybridization, which reduces the indefinite system to a symmetric positive-semidefinite system. The condition number of the resulting system, which is characterized by a non-inherited bilinear form, depends not only on the grid size but also on the material parameters and the complexity of the grids. By constructing uniformly stable interpolation operators between the non-nested spaces, we prove that our multilevel solvers converge uniformly with respect to both the grid size and Poisson's ratio. Numerical experiments are presented to validate our theoretical results.

Motivation
Some biological soft tissues, such as the arterial wall, are nearly incompressible and are reinforced by collagen fibers, which induce the anisotropy in the mechanical response. To obtain physiologically realistic models for such materials, one needs to understand the orientations of the fibers. However, the in-situ identification of patient-specific fiber orientations in real arteries, whose geometry significantly differs from an idealized thick-walled tube, in particular for atherosclerotic arteries, is difficult. Another promising scheme for the automated calculation of the fiber orientations is based on the assumption that the fiber orientations are mainly governed by the principal tensile stress directions resulting in an improved load transfer within the artery. This scheme requires the numerical approximation for the symmetric stress and then the computation of the principal stress.

The mixed finite element methods are popular in mechanics since they avoid locking and provide a straightforward approximation for stress. However, the construction of stable finite element spaces for the classical Hellinger-Reissner variational formulation using polynomial shape functions is very challenging. In the following, we consider the following variational problem: Find $(\sigma, \nu) \in \Sigma \times \nu$ such that
\[
\begin{align*}
\llangle A \nu, \tau \rrangle + (\sigma, \tau) &= 0 \quad \forall \tau \in \Sigma, \\
(\sigma, v) &= (f, v) \quad \forall v \in \nu.
\end{align*}
\]

Main Objectives and Difficulties
- **Objective 1:** High-precision and structure-preserving approaches for stress analysis.
- **Objective 2:** Robust and scalable iterative solvers for nearly incompressible materials.
- **Difficulty 1:** Large system $\Sigma \times \nu$.
  - High-order ($k \geq 1$) conforming elements: Hu-Zhang (2014, 2015)
  - Lowest conforming elements in $3D$: $P_k(\Sigma) \times P_{k-1}(\nu)$, number of local DOF:
  \[ NC_k^2 + N_k^2 = 210 + 60 = 270. \]
- **Difficulty 2:** hard to design solver for mixed formulation.
- **Difficulty 3:** nearly incompressible material.

Mixed Methods and Hybridization
- Discrete stress space and discrete displacement space
- Hybridization: continuous stress space $\mathcal{L}^2$ and multiplier space $\mathcal{P}_k(\nu)$.
- Hybridized mixed: find $(\sigma_k, \nu_k)$ such that
  \[
  \begin{align*}
  (\sigma_k, \nu) &= (f, \nu) \quad \forall \nu \in \nu_k, \\
  (\sigma_k, v_k) &= (f, v) \quad \forall v \in \nu_k.
  \end{align*}
  \]

Theorem 0.1. Let $(\sigma, \nu) \in \Sigma \times \nu$ be the exact solution of the problem (1) and $(\sigma_k, \nu_k) \in \Sigma_k \times \nu_k$ be the finite element solution. Assume that $u \in H^{1+s}(\Omega)$ and $u \in H^{1+s}(\Omega)$. We have
\[
\| \sigma_k - \sigma \|_H^2 \leq \frac{1}{2} \| u \|_{H^{1+s}}^2 + \frac{1}{2} \| u \|_{H^{1+s}}^2.
\]

Theorem 0.2. The Lagrange multiplier $\lambda_k$ satisfies
\[
\lambda_k = 2\lambda + \Delta \theta_k,
\]
where $s \lambda_k = L^2(\Omega)$, Moreover, the system (3) is symmetric positive-semidefinite and its kernel is $G(\mathcal{S})$. We also have the condition number estimate:
\[
cond(S) \leq \frac{2^6 + 1}{2^6} = 128/64 = 2.
\]

Multilevel Solvers
- Choosing a good smoother, appropriate coarse-scale problems and inter-scale transfer operators.
- Constructing coarse-scale approximations to the fine-scale variables.

Schwarz Smoother
- Overlapping subdomains $(\Omega_i)_{i=1}^\beta$ measures the amount of overlap.

\[
\begin{align*}
\text{Subspaces for } i \leq j, M_i &= \{ \lambda \in \mathcal{P}_k(\nu_k) : \lambda = 0 \quad \forall \nu \in \mathcal{P}_k(\nu_k) \}, \\
\text{Subproblems } S_i, M_i &\rightarrow M_j, \text{ where } (\lambda_i, w_i) = (\lambda_i, w_j).
\end{align*}
\]

Conclusions
1. A family of mixed finite element for elasticity.
2. Using hybridization to reduce the dimension of linear system.
3. The solution cost is dominated by solving a SPD system.
4. Two-level and multilevel preconditioner using the primal formulation as the coarse problem.
5. Future works: singular vertex.

References

Numerical Results
- **Convergence tests for the lowest order:**
  - **Table 1:** Errors and observed convergence orders on macro-grid, $k=0$
  - **Table 2:** Errors and observed convergence orders on ecm-grid, $k=2$

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