A Robust Multilevel Solver for a New Hybridizable Mixed Discretization of Linear Elasticity

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Outline

Linear elasticity

- Linear elasticity in stress-displacement formulation:

$$\begin{cases} \mathcal{A}\boldsymbol{\sigma} - \epsilon(u) = 0 & \text{in } \Omega \subset \mathbb{R}^d, \\ \mathrm{div}\boldsymbol{\sigma} = -f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

displacement: $u : \Omega \mapsto \mathbb{R}^n$ $\quad$ $\mathcal{A} : \mathbb{S} \mapsto \mathbb{S}$: compliance operator

stress: $\boldsymbol{\sigma} : \Omega \mapsto \mathbb{S} := \mathbb{R}^{n\times n}_{sym}$ $\quad$ $\epsilon(u) := (\nabla u + (\nabla u)^T)/2$

- Constitutive law: $\boldsymbol{\sigma} = 2\tilde{\mu}\epsilon(u) + \tilde{\lambda}\mathrm{div} u \boldsymbol{I}$

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\tilde{\mu}}\left(\boldsymbol{\sigma} - \frac{\tilde{\lambda}}{2\tilde{\mu} + d\tilde{\lambda}}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right) \rightarrow \frac{1}{2\tilde{\mu}}(\boldsymbol{\sigma} - \frac{1}{d}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}) \quad \text{as } \tilde{\lambda} \rightarrow \infty. \tag{2}$$

- Lamé constants: $\tilde{\mu} = \mathcal{O}(1)$, $\tilde{\lambda} \gg 1$ for nearly incompressible materials.

BDM-type elements for symmetric tensor

- Hellinger-Reissner principle:

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega + (u, \mathrm{div}\boldsymbol{\tau})_\Omega = 0 & \forall \boldsymbol{\tau} \in H(\mathrm{div}; \mathbb{S}), \\ (\mathrm{div}\boldsymbol{\sigma}, v)_\Omega = -(f, v)_\Omega & \forall v \in L^2(\mathbb{R}^d). \end{cases} \tag{3}$$

- Discrete stress space: normal continuity on faces

$$\Sigma_{h,k+1} = \{\boldsymbol{\tau} \in H(\mathrm{div}, \mathbb{S}) \mid \boldsymbol{\tau}|_K \in P_{k+1}(K; \mathbb{S}) \quad \forall K \in \mathcal{T}_h\}.$$

Discrete displacement space:

$$V_{h,k} = \{v \in L^2(\mathbb{R}^d) \mid v|_K \in P_k(K, \mathbb{R}^d)\}.$$

- Mixed method: Find $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_{h,k+1} \times V_{h,k}$ such that

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau})_\Omega + (u_h, \mathrm{div}\boldsymbol{\tau})_\Omega = 0 & \forall \boldsymbol{\tau} \in \Sigma_{h,k+1}, \\ (\mathrm{div}\boldsymbol{\sigma}_h, v)_\Omega = -(f, v)_\Omega & \forall v \in V_{h,k}. \end{cases} \tag{4}$$

Comments on mixed method for linear elasticity

- Locking-free scheme, suitable stress analysis ☺
- High order ($k \geq d$) conforming elements: Hu-Zhang (2014, 2015)
- Difficulty 1: large system ☹
  - No low order ($k \leq d-1$) conforming elements: Wu-Xu-Gong (2015)
  - Lowest conforming elements in 2D: $\mathcal{P}_3(\mathbb{S}) - \mathcal{P}_2(\mathbb{R}^2)$, number of local d.o.f.
  $$3C_5^2 + 2C_4^2 = 30 + 12 = 42.$$
  - Lowest conforming elements in 3D: $\mathcal{P}_4(\mathbb{S}) - \mathcal{P}_3(\mathbb{R}^3)$, number of local d.o.f.
  $$6C_7^3 + 3C_6^3 = 210 + 60 = 270.$$
- Difficulty 2: hard to design solver for mixed formulation ☹
- Difficulty 3: nearly incompressible material ☹

Hybridized mixed method

- Hybridization for Poisson: Arnold-Brezzi (1985), Brezzi-Douglas-Marini (1985), Brezzi-Douglas-Duran-Fortin (1987) ...

$H(\mathrm{div}, \mathbb{S}) \Longleftrightarrow L^2(\mathbb{S}) +$ Lagrange multiplier on the normal component

- Discontinuous stress space + Lagrange multiplier space:

$$\Sigma_{h,k+1}^{-1} = \{\boldsymbol{\tau}_h \in L^2(\mathbb{S}) \mid \boldsymbol{\tau}_h|_K \in P_{k+1}(K;\mathbb{S}) \quad \forall K \in \mathcal{T}_h\}.$$

$$M_{h,k+1} = \{\mu_h \in L^2(\mathcal{F}_h;\mathbb{R}^d) \mid \mu_h|_F = P_{k+1}(F,\mathbb{R}^d) \quad \forall F \in \mathcal{F}_h^i \text{ and } \mu|_{\mathcal{F}_h^\partial} = 0\}$$

- Hybridized mixed method: find $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \Sigma_{h,k+1}^{-1} \times V_{h,k} \times M_{h,k+1}$ such that

$$(\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\mathrm{div}\boldsymbol{\tau}_h, u_h) + \langle [\boldsymbol{\tau}_h], \lambda_h \rangle_{\mathcal{F}_h^i} = 0 \qquad \forall \boldsymbol{\tau}_h \in \Sigma_{h,k+1}^{-1}, \tag{5a}$$

$$(\mathrm{div}\boldsymbol{\sigma}_h, v_h) = -(f, v_h) \qquad \forall v_h \in V_{h,k}, \tag{5b}$$

$$\langle [\boldsymbol{\sigma}_h], \mu_h \rangle_{\mathcal{F}_h^i} = 0 \qquad \forall \mu_h \in M_{h,k+1}. \tag{5c}$$

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$$M_{h,k+1} = \{\mu_h \in L^2(\mathcal{F}_h;\mathbb{R}^d) \mid \mu_h|_F = P_{k+1}(F,\mathbb{R}^d) \quad \forall F \in \mathcal{F}_h^i \text{ and } \mu|_{\mathcal{F}_h^\partial} = 0\}$$

- Hybridized mixed method: find $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \Sigma_{h,k+1}^{-1} \times V_{h,k} \times M_{h,k+1}$ such that

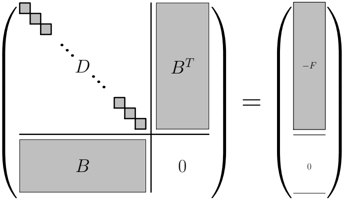
$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\mathrm{div}\boldsymbol{\tau}_h, u_h) + \langle [\boldsymbol{\tau}_h], \lambda_h \rangle_{\mathcal{F}_h^i} = 0 && \forall \boldsymbol{\tau}_h \in \Sigma_{h,k+1}^{-1}, && (5a)\\
&(\mathrm{div}\boldsymbol{\sigma}_h, v_h) = -(f, v_h) && \forall v_h \in V_{h,k}, && (5b)\\
&\langle [\boldsymbol{\sigma}_h], \mu_h \rangle_{\mathcal{F}_h^i} = 0 && \forall \mu_h \in M_{h,k+1}. && (5c)
\end{aligned}
$$

- Key: Linear system becomes larger, but easier to solve. Why?

Matrix form for λ_h

$$\left(\begin{array}{cc|c} (\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) & (\mathrm{div}\boldsymbol{\tau}_h, u_h) & \multirow{2}{*}{$\langle [\boldsymbol{\tau}_h], \lambda_h \rangle$} \\ (\mathrm{div}\boldsymbol{\sigma}_h, v_h) & 0 & \\ \hline \multicolumn{2}{c|}{\langle [\boldsymbol{\sigma}_h], \mu_h \rangle} & 0 \end{array}\right) = \left(\begin{array}{c} -(f, v_h) \\ \hline 0 \end{array}\right).$$

The matrix form looks like:



D is a block-diagonal matrix.

- Step 1: Solve λ_h by Schur complement: (much smaller SPSD system)

$$(BD^{-1}B^T)\lambda_h = -BD^{-1}F. \tag{6}$$

- Step 2: Recover the stress and displacement locally:

$$\boldsymbol{\sigma}_h, u_h \leftarrow D^{-1}(-F - B^T\lambda_h). \tag{7}$$

Charaterization of $BD^{-1}B^T\lambda = -BD^{-1}F$

- Local solver: For any $\lambda \in M_{h,k+1}$, define $(\boldsymbol{\sigma}_\lambda, u_\lambda) \in \Sigma_{h,k+1}^{-1} \times V_{h,k}$ by

$$(\mathcal{A}\boldsymbol{\sigma}_\lambda, \boldsymbol{\tau}_h)_K + (u_\lambda, \mathrm{div}\boldsymbol{\tau}_h)_K = \langle \lambda, \boldsymbol{\tau}_h n\rangle_{\partial K} \qquad \forall \boldsymbol{\tau}_h \in \Sigma_{h,k+1}^{-1}, \tag{8a}$$

$$(\mathrm{div}\boldsymbol{\sigma}_\lambda, v_h)_K = 0 \qquad \forall v_h \in V_{h,k}. \tag{8b}$$

$$\boldsymbol{\sigma}_\lambda, u_\lambda \leftarrow -D^{-1}B^T\lambda, \qquad \boldsymbol{\sigma}_\lambda|_K \in Z_h(K) := \{\boldsymbol{\tau}_h \in \Sigma_{h,k+1}^{-1}(K) : \mathrm{div}\boldsymbol{\tau}_h = 0\}.$$

- Bilinear form of LHS:

$$\begin{aligned} s(\lambda,\mu) &:= (BD^{-1}B^T\lambda, \mu) = (D(-D^{-1}B^T\lambda), (-D^{-1}B^T)\mu) \\ &= (D\begin{pmatrix}\boldsymbol{\sigma}_\lambda \\ u_\lambda\end{pmatrix}, \begin{pmatrix}\boldsymbol{\sigma}_\mu \\ u_\mu\end{pmatrix}) = (\mathcal{A}\boldsymbol{\sigma}_\lambda, \boldsymbol{\sigma}_\mu)_\Omega. \end{aligned} \tag{9}$$

- Linear form of RHS:

$$(-BD^{-1}F, \mu) = (F, \begin{pmatrix}\boldsymbol{\sigma}_\mu \\ u_\mu\end{pmatrix}) = (f, u_\mu).$$

- Variational formulation: find $\lambda \in M_{h,k+1}$ such that

$$s(\lambda,\mu) := (\mathcal{A}\boldsymbol{\sigma}_\lambda, \boldsymbol{\sigma}_\mu) = (f, u_\mu), \qquad \forall \mu \in M_{h,k+1}. \tag{10}$$

Difficulty in nearly incompressible case

- Recall the difficulties: ~~large system~~, ~~mixed formulation~~, nearly incompressible materials.

$$(S\lambda, \mu) := (\mathcal{A}\boldsymbol{\sigma}_\lambda, \boldsymbol{\sigma}_\mu) = (f, u_\mu), \qquad \forall \mu \in M_{h,k+1}. \tag{11}$$

- Question: is it uniformly convergent when Lamé constant $\tilde{\lambda} \to \infty$?

Example (Gauss-Seidel on 2×2 uniform grid)

- Lagrange multiplier: $\mathcal{P}_3$, 64×64 matrix
- Number of iterations:

$\tilde{\lambda}$	Number of iterations
10^0	127
10^1	142
10^2	591
10^3	5099
10^4	39267
10^5	271122

Near incompressibility $\Longleftrightarrow$ nearly singular system

- Nearly singular in compliance tensor $\mathcal{A}$:

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\tilde{\mu}}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\tilde{\mu} + d\tilde{\lambda}}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right) \rightarrow \frac{1}{2\tilde{\mu}}(\boldsymbol{\sigma} - \frac{1}{d}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}) \quad \text{as } \tilde{\lambda} \rightarrow \infty.$$

- Nearly singular in λ: taking $\boldsymbol{\tau} = \boldsymbol{I}$ in the definition of $(\boldsymbol{\sigma}_\lambda, u_\lambda)$:

$$(\mathcal{A}\boldsymbol{\sigma}_\lambda, \boldsymbol{\tau})_K + (\mathrm{div}\boldsymbol{\tau}, u_\lambda) = \langle \lambda, \boldsymbol{\tau} n\rangle \quad \Rightarrow \int_{\partial K} \lambda \cdot n\, ds \rightarrow 0 \quad \text{as } \tilde{\lambda} \rightarrow \infty.$$

Define a semi-norm

$$|\lambda|_{*,K} = |K|^{-1/2} \left| \int_{\partial K} \lambda \cdot n\, ds \right|.$$

Motivated from the Local solver (8), we define another semi-norm

$$|\lambda|_{h,K} := \sup_{\boldsymbol{\tau}_h \in Z_h(K), \boldsymbol{\tau}_h \neq 0} \frac{\langle \lambda, \boldsymbol{\tau}_h n\rangle_{\partial K}}{\|\boldsymbol{\tau}_h\|_{0,K}}.$$

Lemma (G., Wu, Xu, 2016)

$$\|\lambda\|_S^2 \eqsim 2\tilde{\mu}|\lambda|_h^2 + \tilde{\lambda}|\lambda|_*^2 \qquad \forall \lambda \in M_{h,k+1}. \tag{12}$$

Condition number estimate

- Upper bound:

$$\|\lambda\|_S^2 \lesssim (2\tilde{\mu} + \tilde{\lambda}) h^{-1} \|\lambda\|_0^2 \qquad \forall \lambda \in M_{h,k+1}. \tag{13}$$

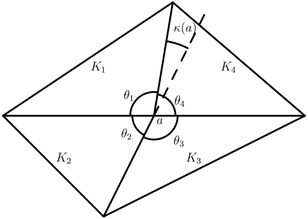
Condition number estimate

- Upper bound:

$$\|\lambda\|_S^2 \lesssim (2\tilde{\mu} + \tilde{\lambda}) h^{-1} \|\lambda\|_0^2 \qquad \forall \lambda \in M_{h,k+1}. \tag{13}$$

- Lower bound: depends on the singular vertices. Define

$$\kappa(a) = \max\{|\theta_i + \theta_j - \pi| \mid 1 \le i,j \le m \text{ and } i - j = 1 \mod m\}.$$



$$2\tilde{\mu} h \sin^2(\kappa_0) \|\lambda\|_0^2 \lesssim \|\lambda\|_S^2 \qquad \forall \lambda \in M_{h,k+1}. \tag{14}$$

- Condition number:

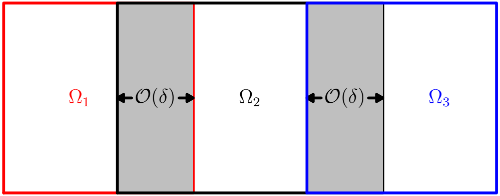
$$\mathrm{cond}(S) \lesssim \frac{2\tilde{\mu} + \tilde{\lambda}}{2\tilde{\mu}} h^{-2} \sin^{-2}(\kappa_0)$$

Construction of multilevel methods

- Choosing a good smoother.
- Choosing appropriate coarse-scale problems.
- Choosing inter-scale transfer operators.
- Constructing coarse-scale approximations to the fine-scale variables.

Schwarz smoother

- Overlapping subdomains $\{\Omega_i\}_{i=1}^J$, δ measures the amount of overlap.



- Subspaces for $1 \le i \le J$,

$$M_i = \{\lambda \in M_{h,k+1} \mid \lambda|_F = 0 \;\; \forall F \in \mathcal{F}_h \backslash \Omega_i^0\}.$$

- $S_i : M_i \mapsto M_i'$, where $\langle S_i \lambda_i, \mu_i \rangle := s(\iota_i \lambda_i, \iota_i \mu_i)$
- Partition of unity:

$$\theta_i = 0 \qquad \text{on } \Omega \setminus \Omega_i, \tag{15a}$$

$$\sum_{i=1}^{J} \theta_i = 1 \qquad \text{on } \bar{\Omega}, \tag{15b}$$

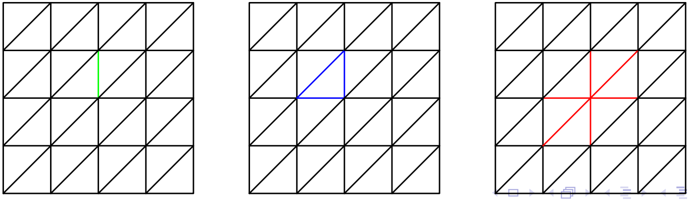
$$\|\nabla \theta_i\|_\infty \lesssim \delta^{-1}. \tag{15c}$$

Minimal requirement of subdomains

- Kernel preserving decomposition: each basis of the ND_0 should at least in one subdomain.
- 2D case: $ND_0 = \mathcal{P}_1$ Lagrange element $\Rightarrow$ point-patch supported

Subdomains \ $\tilde{\lambda}$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8
Edges	31	36	59	79	109	131	154	181	211
Elements	12	15	24	33	45	54	62	72	82
Point Patches	8	10	12	13	13	14	14	14	14

Table: PCG, Multiplicative Schwarz smoother, uniform grid with size $h = 1/4$, $tol = 1e - 6$.



Coarse problem

- Revisit the norm equivalence

$$\|\lambda\|_S^2 \eqsim 2\tilde{\mu}|\lambda|_h^2 + \tilde{\lambda}|\lambda|_*^2 \qquad \forall \lambda \in M_{h,k+1}.$$

- Comparing to the primal elasticity

$$\|w_H\|_{A_H}^2 \eqsim 2\tilde{\mu}|\epsilon(w_H)|_0^2 + \tilde{\lambda}\|P_0^H \mathrm{div}\, w_H\|_0^2,$$

with

$$\langle A_H w_H, v_H \rangle = 2\tilde{\mu}(\epsilon(w_H), \epsilon(v_H)) + \tilde{\lambda}(P_0^H \mathrm{div}\, w_H, P_0^H \mathrm{div}\, v_H),$$

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- Revisit the norm equivalence

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$$\|w_H\|_{A_H}^2 \eqsim 2\tilde{\mu}|\epsilon(w_H)|_0^2 + \tilde{\lambda}\|P_0^H \mathrm{div} w_H\|_0^2,$$

with

$$\langle A_H w_H, v_H\rangle = 2\tilde{\mu}(\epsilon(w_H), \epsilon(v_H)) + \tilde{\lambda}(P_0^H \mathrm{div} w_H, P_0^H \mathrm{div} v_H),$$

- Key idea: Using the Lagrange element $\mathcal{P}_2$ as the coarse space.

$$W_H := \{w \in H_0^1(\Omega;\mathbb{R}^2) \mid w|_K \in \mathcal{P}_2(K;\mathbb{R}^2) \text{ for } K \in \mathcal{T}_H\}.$$

A two-level additive Schwarz preconditioner is

$$B = I_H^h A_H^{-1}(I_H^h)' + \sum_{i=1}^{J} \iota_i S_i^{-1} \iota_i'. \tag{16}$$

Question: $I_H^h : W_H \mapsto M_h$?

Intergrid transfer operator

1. Step 1: Harmonic extension: the harmonic extension $\tilde{I}_H^h : W_H \mapsto W_h$ (Schöberl, 1999). On each edge of coarse element $K_H \in \mathcal{T}_H$

$$\begin{aligned} \tilde{I}_H^h w_H|_{\partial K_H} &= w_H|_{\partial K_H}, \\ a_h(\tilde{I}_H^h w_H, v_h) &= 0 \qquad \forall v_h \in W_{h,0}(K_H). \end{aligned} \tag{17}$$

Property of $\tilde{I}_H^h$: $\|\tilde{I}_H^h w_H\|_{A_h} \lesssim \|w_H\|_{A_H}$.

2. Step 2: $Q_h : W_h \mapsto M_{h,k+1}$, L^2 projection.

The intergrid transfer operator I_H^h appearing in (16):

$$I_H^h := Q_h \tilde{I}_H^h : W_H \mapsto M_{h,k+1}. \tag{18}$$

Stability of intergrid transfer operator

For any w_H, let $w_h = \tilde{I}_H^h w_H$,

- Nearly incompressible part

$$|Q_h w_h|_{*,K} = |K|^{-1/2}\left|\int_{\partial K} Q_h w_h \cdot \nu ds\right| = |K|^{-1/2}\left|\int_{\partial K} w_h \cdot \nu ds\right|$$
$$= |K|^{-1/2}\left|\int_K \mathrm{div}\, w_h dx\right| = \|P_0^h \mathrm{div}\, w_h\|_{0,K},$$

- Other part

$$|Q_h w_h|_{h,K} = \sup_{\boldsymbol{\tau}\in Z_h(K)} \frac{\langle Q_h w_h, \boldsymbol{\tau}\nu\rangle_{\partial K}}{\|\boldsymbol{\tau}\|_{0,K}} = \sup_{\boldsymbol{\tau}\in Z_h(K)} \frac{\langle w_h, \boldsymbol{\tau}\nu\rangle_{\partial K}}{\|\boldsymbol{\tau}\|_{0,K}}$$
$$= \sup_{\boldsymbol{\tau}\in Z_h(K)} \frac{(\epsilon(w_h), \boldsymbol{\tau})_K}{\|\boldsymbol{\tau}\|_{0,K}} \leq \|\epsilon(w_h)\|_{0,K},$$

Therefore,

$$\|I_H^h w_H\|_S^2 \eqsim 2\tilde{\mu}|Q_h w_h|_h^2 + \tilde{\lambda}|Q_h w_h|_*^2 \lesssim \|w_h\|_{A_h}^2 \lesssim \|w_H\|_{A_H}^2.$$

Coarse Approximation: $\Pi_h : M_{h,k+1} \mapsto W_h$

- Parameter-independent problem: finding $(\bar{\sigma}_\lambda, \bar{u}_\lambda) \in \Sigma^{-1}_{h,k+1} \times V_{h,k}$ such that, for every element $K \in \mathcal{T}_h$,

$$(\bar{\sigma}_\lambda, \tau_h)_K + (\bar{u}_\lambda, \mathrm{div}\tau_h)_K = \langle \lambda, \tau_h \nu \rangle_{\partial K}, \qquad \forall \tau_h \in \Sigma_{h,k+1}(K), \qquad (19a)$$

$$(\mathrm{div}\bar{\sigma}_\lambda, v_h)_K = 0, \qquad \forall v_h \in V_{h,k}(K). \qquad (19b)$$

- Projection $P_{K,rm} : M_{h,k+1}(\partial K) \mapsto RM(K)$ by

$$(P_{K,rm}\lambda, r)_K = (\bar{u}_\lambda, r)_K \quad \forall r \in RM(K).$$

1. Step 1: Clément type interpolation $\Pi_{1,h} : M_{h,k+1} \mapsto \mathcal{P}_{1,h} \cap H^1(\Omega; \mathbb{R}^2)$

$$(\Pi_{1,h}\lambda)(a) := \begin{cases} \frac{\sum_{K\in\omega_a}(P_{K,rm}\lambda)(a)}{\sum_{K\in\omega_a} 1} & \text{otherwise,} \\ 0 & a \in \partial\Omega. \end{cases}$$

2. Step 2: correction operator $\Pi_{2,h} : M_{h,k+1} \cup H_1(\Omega, \mathbb{R}^2) \mapsto W_h$:

$$(\Pi_{2,h}\lambda)(a) := 0 \quad \forall a \in \mathcal{N}_h \quad \text{and} \quad \int_F I_{2,h}\lambda \, ds := \int_F \lambda \, ds \quad \forall F \in \mathcal{F}_h.$$

3. I_h is composed by these two operators, for any $\lambda \in M_{h,k+1}$,

$$\Pi_h\lambda := \Pi_{1,h}\lambda + \Pi_{2,h}(\lambda - \Pi_{1,h}\lambda). \qquad (20)$$

Coarse approximation

Lemma (G., Wu, Xu, 2016)

For any $\lambda \in M_{h,k+1}$, it holds that

$$\begin{aligned} \int_F \Pi_h \lambda &= \int_F \lambda, \\ \|\Pi_h \lambda\|_{A_h} &\lesssim \|\lambda\|_S, \\ \|\lambda - Q_h \Pi_h \lambda\|_0^2 &\lesssim h \|\lambda\|_S^2. \end{aligned} \tag{21}$$

Coarse approximation

Lemma (G., Wu, Xu, 2016)

For any $\lambda \in M_{h,k+1}$, it holds that

$$\int_F \Pi_h \lambda = \int_F \lambda, \quad \|\Pi_h\lambda\|_{A_h} \lesssim \|\lambda\|_S, \quad \|\lambda - Q_h\Pi_h\lambda\|_0^2 \lesssim h\|\lambda\|_S^2. \tag{21}$$

$$\Uparrow$$

Lemma (G., Wu, Xu, 2016)

Assuming $\kappa \geq \kappa_0 > 0$ and the girds are shape regular, it holds that

$$h^{-1}\|\lambda - P_{K,rm}\lambda\|_{0,\partial K}^2 \lesssim \sin(\kappa_0)^{-1} \sum_{K' \in \omega_K} |\lambda|_{h,K'}^2, \tag{22}$$

where ω_K denotes the set of all elements that share vertex with K.

Stable decomposition

Theorem (G., Wu, Xu, 2016)

For any $\lambda \in M_{h,k+1}$, there exists a decomposition $\lambda = I_H^h w_H + \sum_{i=1}^J \lambda_i$ such that $w_H \in W_H, \lambda_i \in M_i$, and

$$\|w_H\|_{A_H}^2 + \sum_{i=1}^J \|\lambda_i\|_S^2 \lesssim \frac{H^2}{\delta^2} \|\lambda\|_S^2. \quad (23)$$

Sketch of the proof: $Q_0^\perp$ is the L^2 projection on $M_{h,0}^\perp$

$$\begin{aligned}
\lambda &= Q_h \underbrace{\Pi_h \lambda}_{w_h} + \underbrace{(I - Q_h \Pi_h)\lambda}_{\lambda_0} \\
&= Q_h(\tilde{I}_H^h w_H + \sum_{i=1}^J w_i) + \sum_{i=1}^J Q_0^\perp(\theta_i \lambda_0) \qquad (\text{Schöberl, 1999}) \\
&= I_H^h w_H + \sum_{i=1}^J \underbrace{Q_h w_i + Q_0^\perp(\theta_i \lambda_0)}_{\lambda_i}
\end{aligned}$$

Stable decomposition II

From stable decomposition between W_h and W_H,

$$\|w_H\|_{A_H}^2 + \sum_{i=1}^{J} \|Q_h w_i\|_S^2 \lesssim \|w_H\|_{A_H}^2 + \sum_{i=1}^{J} \|w_i\|_{A_h}^2 \lesssim \frac{H^2}{\delta^2} \|w_h\|_{A_h}^2,$$

$$\begin{aligned}
\sum_{i=1}^{J} \|Q_0^{\perp}(\theta_i \lambda_0)\|_S^2 &= \sum_{i=1}^{J} \sum_{K \in \mathcal{T}_h \cap \Omega_i} \|Q_0^{\perp}(\theta_i \lambda_0)\|_{S,K}^2 \\
&\lesssim \sum_{i=1}^{J} \sum_{K \in \mathcal{T}_h \cap \Omega_i} h_K^{-1} \|Q_0^{\perp}(\theta_i \lambda_0)\|_{0,\partial K}^2 \\
&\lesssim \sum_{i=1}^{J} \sum_{K \in \mathcal{T}_h \cap \Omega_i} h_K^{-1} \|\lambda_0\|_{0,\partial K}^2 \\
&\lesssim \|\lambda\|_S^2. \qquad \square
\end{aligned}$$

Multigrid preconditioner: $A_H^{-1} \approx \tilde{B}_H$ (Schöberl, 1999)

$$B = I_H^h \tilde{B}_H (I_H^h)' + \sum_{i=1}^{J} \iota_i S_i^{-1} \iota_i'.$$

Numerical results

- The Lamé constants are set as $\tilde{\mu} = 1/2$ and $\tilde{\lambda} = \frac{\tilde{\nu}}{1-2\tilde{\nu}}$.
- Two-level additive Schwarz preconditioner, $H/\delta = 2$

$1/h$ \ $\tilde{\nu}$	0.49	0.499	0.4999	0.49999	0.499999	0.4999999
4	17	18	21	23	23	23
8	17	20	25	27	28	29
16	18	20	26	28	29	29
32	18	20	25	27	28	29

- Multigrid preconditioner + W-2-2 cycle

$1/h$ \ $\tilde{\nu}$	0.49	0.499	0.4999	0.49999	0.499999	0.4999999
4	4	5	5	5	5	5
8	4	6	7	7	7	7
16	5	6	7	7	7	7
32	5	6	7	7	7	7

Concluding remarks

1. A family of hybridizable mixed finite element for elasticity,
2. The solution cost is dominated by solving a SPD system,
3. Two-level and multilevel precoditioner using the primal formulation as the coarse problem.
4. Future works: singular vertex.

THANK YOU!

Near kernel $|\lambda|_* = 0$ ($|\lambda|_{*,K} = |K|^{-1/2}|\int_{\partial K} \lambda \cdot n|$)

- Kernel-preserving decomposition for nearly singular system: Lee-Wu-Xu-Zikatanov (2007, 2008)
- Key observation: the d.o.f of lowest order Raviart-Thomas is $\int_F w \cdot n\, ds$!
- Surjective linear mapping $\Phi_h : M_{h,k+1} \mapsto RT_0(\mathcal{T}_h)$:

$$\int_F \Phi_h(\lambda) \cdot n\, ds := \int_F \lambda \cdot n\, ds \qquad \forall F \in \mathcal{F}_h^i.$$

- Local basis of near kernel:

$$|\lambda|_* = 0 \quad \Longleftrightarrow \quad \mathrm{div}\Phi_h(\lambda) = 0.$$

$$\mathrm{div}\text{-kernel of } RT_0 \Leftrightarrow \mathrm{curl} ND_0.$$