A Hybridizable Mixed Finite Element Method for Planar Linear Elasticity

Shihua Gong
Joint Work with Prof. Jinchao Xu and Shuonan Wu

Beijing International Center for Mathematical Research, Peking University
gongshihua@pku.edu.cn

Beijing, Aug. 13, 2015

Outline

1. Mixed Methods for Linear Elasticity
2. A Nodal Basis for the Space $H(\mathrm{div}, \mathbb{S}) \cap \mathcal{P}_{k+1}$
3. A Mixed Finite Element Method
4. Hybridization
5. Numerical Results

Outline

1. Mixed Methods for Linear Elasticity
2. A Nodal Basis for the Space $H(\mathrm{div}, \mathbb{S}) \cap \mathcal{P}_{k+1}$
3. A Mixed Finite Element Method
4. Hybridization
5. Numerical Results

Linear Elasticity

We consider the following linear elasticity problem with pure displacement boundary condition

$$\begin{cases} \mathcal{A}\boldsymbol{\sigma} - \epsilon(u) = 0, & \text{in } \Omega, \\ \mathrm{div}\boldsymbol{\sigma} = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^2$.

- displacement: $u : \Omega \mapsto \mathbb{R}^2$
- stress: $\boldsymbol{\sigma} : \Omega \mapsto \mathbb{S} := \mathbb{R}^{2\times 2}_{sym}$
- $\mathcal{A} : \mathbb{S} \mapsto \mathbb{S}$: SPD operator
- $\epsilon(u) := (\nabla u + (\nabla u)^T)/2$

Linear Elasticity

We consider the following linear elasticity problem with pure displacement boundary condition

$$\begin{cases} \mathcal{A}\boldsymbol{\sigma} - \epsilon(u) = 0, & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{\sigma} = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$.

- displacement: $u : \Omega \mapsto \mathbb{R}^2$
- stress: $\boldsymbol{\sigma} : \Omega \mapsto \mathbb{S} := \mathbb{R}^{2\times 2}_{sym}$
- $\mathcal{A} : \mathbb{S} \mapsto \mathbb{S}$: SPD operator
- $\epsilon(u) := (\nabla u + (\nabla u)^T)/2$

Symmetric matrix:

$$\begin{pmatrix} * & \star \\ \star & * \end{pmatrix} \in \mathbb{R}^{2\times 2}_{sym},$$

Hellinger-Reissner Variational Principle

The saddle point problem is to find $(\boldsymbol{\sigma}, u) \in \Sigma \times V$, such that

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega + (\mathrm{div}\boldsymbol{\tau}, u)_\Omega & = 0, \qquad \forall \boldsymbol{\tau} \in \Sigma, \\ (\mathrm{div}\boldsymbol{\sigma}, v)_\Omega & = (f, v)_\Omega, \qquad \forall v \in V. \end{cases} \tag{2}$$

where

$$\Sigma \times V \triangleq H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^2) \tag{3}$$

$$H(\mathrm{div}, \Omega; \mathbb{S}) = \{\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{S}) \mid \mathrm{div}\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{R}^2)\} \tag{4}$$

with the norm defined by

$$\|\boldsymbol{\tau}\|^2_{\mathrm{div},\Omega} \triangleq \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathrm{div}\boldsymbol{\tau}\|^2_{0,\Omega}, \quad \forall \boldsymbol{\tau} \in H(\mathrm{div}, \Omega; \mathbb{S}).$$

Hellinger-Reissner Variational Principle

The saddle point problem is to find $(\boldsymbol{\sigma}, u) \in \Sigma \times V$, such that

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega + (\mathrm{div}\boldsymbol{\tau}, u)_\Omega &= 0, & \forall \boldsymbol{\tau} \in \Sigma, \\ (\mathrm{div}\boldsymbol{\sigma}, v)_\Omega &= (f, v)_\Omega, & \forall v \in V. \end{cases} \tag{2}$$

where

$$\Sigma \times V \triangleq H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^2) \tag{3}$$

$$H(\mathrm{div}, \Omega; \mathbb{S}) = \{\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{S}) \mid \mathrm{div}\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{R}^2)\} \tag{4}$$

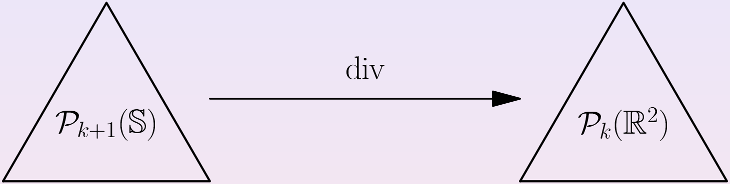
with the norm defined by

$$\|\boldsymbol{\tau}\|^2_{\mathrm{div},\Omega} \triangleq \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathrm{div}\boldsymbol{\tau}\|^2_{0,\Omega}, \quad \forall \boldsymbol{\tau} \in H(\mathrm{div}, \Omega; \mathbb{S}).$$

- Avoid Locking phenomena
- Provide direct approximation to stress

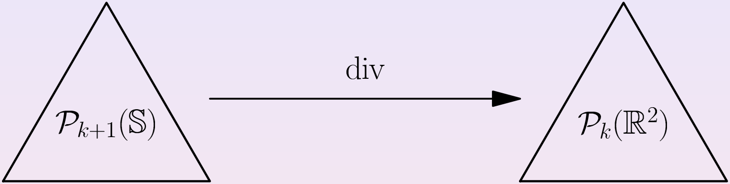
Requirements for the Finite Elements of Stress

- Natural discretization: $\mathcal{P}_{k+1} - \mathcal{P}_k^{-1}$



Requirements for the Finite Elements of Stress

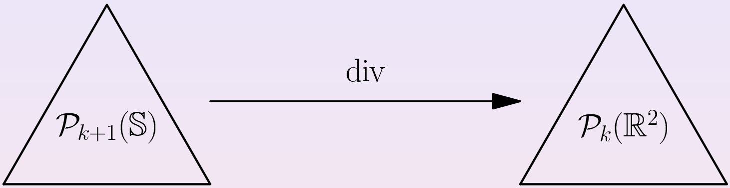
- Natural discretization: $\mathcal{P}_{k+1} - \mathcal{P}_k^{-1}$



- Conservation of angular momentum:
  - `the stress tensor is required to be symmetric.`

Requirements for the Finite Elements of Stress

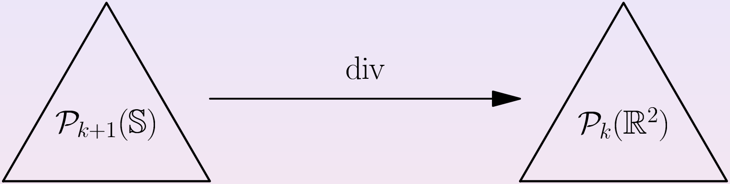
- Natural discretization: $\mathcal{P}_{k+1} - \mathcal{P}_k^{-1}$



- Conservation of angular momentum:
  - `the stress tensor is required to be symmetric.`
- Conservation of translational momentum:
  - `the stress approximation must lie in` $H(\mathrm{div})$.

Requirements for the Finite Elements of Stress

- Natural discretization: $\mathcal{P}_{k+1} - \mathcal{P}_k^{-1}$



- Conservation of angular momentum:
  - the stress tensor is required to be symmetric.
- Conservation of translational momentum:
  - the stress approximation must lie in $H(\mathrm{div})$.

Two requirements for the stress element: symmetric and conforming!

Literature Review

- Composite Element: different grids for stress and displacement;
  - Johnson-Mercier(1978), Arnold-Douglas(1984),
- Weakly Symmetric and Conforming Element: introducing a Lagrangian multipiler to enforce stress weakly symmetric;
  - Amara-Thomas(1979), Stenberg(1988), Farhloul-Fortin(1997), Qiu-Demkowicz(2009)
- Symmetric and non-Conforming Element: relaxing the conformity.
  - Arnold-Winther(2003), Yi(2005,2006), Hu-Shi(2007), Gopalakrishnan-Guzman(2011), Arnold-Awanou-Winther(2014), Gong-Wu-Xu(2015)

Literature Review

- Composite Element: different grids for stress and displacement;
  - `Johnson-Mercier(1978), Arnold-Douglas(1984),`
- Weakly Symmetric and Conforming Element: introducing a Lagrangian multipiler to enforce stress weakly symmetric;
  - `Amara-Thomas(1979), Stenberg(1988), Farhloul-Fortin(1997), Qiu-Demkowicz(2009)`
- Symmetric and non-Conforming Element: relaxing the conformity.
  - `Arnold-Winther(2003), Yi(2005,2006), Hu-Shi(2007), Gopalakrishnan-Guzman(2011), Arnold-Awanou-Winther(2014), Gong-Wu-Xu(2015)`
- Symmetric and Conforming Element:
  - `Arnold-Winther(2002), Adams-Cockburn(2005), Arnold-Awanou-Winther(2008), Hu-Zhang(2014, 2015)`

Literature Review

- Composite Element: different grids for stress and displacement;
  - Johnson-Mercier(1978), Arnold-Douglas(1984),
- Weakly Symmetric and Conforming Element: introducing a Lagrangian multipiler to enforce stress weakly symmetric;
  - Amara-Thomas(1979), Stenberg(1988), Farhloul-Fortin(1997), Qiu-Demkowicz(2009)
- Symmetric and non-Conforming Element: relaxing the conformity.
  - Arnold-Winther(2003), Yi(2005,2006), Hu-Shi(2007), Gopalakrishnan-Guzman(2011), Arnold-Awanou-Winther(2014), Gong-Wu-Xu(2015)
- Symmetric and Conforming Element:
  - Arnold-Winther(2002), Adams-Cockburn(2005), Arnold-Awanou-Winther(2008), Hu-Zhang(2014, 2015)

Question: symmetric, conforming and hybridizable elements?

Outline

1 Mixed Methods for Linear Elasticity

2 A Nodal Basis for the Space $H(\mathrm{div}, \mathbb{S}) \cap \mathcal{P}_{k+1}$

3 A Mixed Finite Element Method

4 Hybridization

5 Numerical Results

The Discrete Stress Space

Our discrete stress space is defined as follows.

$$\Sigma_{h,k+1} = \{\tau \in H(\Omega; \mathrm{div}, \mathbb{S}) \mid \tau|_K \in P_{k+1}(K; \mathbb{S}) \quad \forall K \in \mathcal{T}_h\}. \tag{5}$$

That is the full $C^{\mathrm{div}} - P_{k+1}$ space.

The Discrete Stress Space

Our discrete stress space is defined as follows.

$$\Sigma_{h,k+1} = \{\boldsymbol{\tau} \in H(\Omega; \mathrm{div}, \mathbb{S}) \mid \boldsymbol{\tau}|_K \in P_{k+1}(K; \mathbb{S}) \quad \forall K \in \mathcal{T}_h\}. \tag{5}$$

That is the full $C^{\mathrm{div}} - P_{k+1}$ space.

Using Ciarlet's notation, a finite element is usually defined by a triple $(\mathcal{T}, \mathcal{P}_K, \mathcal{L}_K)$.

- A triangulation $\mathcal{T}$ consisting of polyhedral elements K;
- For each K, a space of shape function $\mathcal{P}_K$;

The Discrete Stress Space

Our discrete stress space is defined as follows.

$$\Sigma_{h,k+1} = \{\boldsymbol{\tau} \in H(\Omega; \mathrm{div}, \mathbb{S}) \mid \boldsymbol{\tau}|_K \in P_{k+1}(K; \mathbb{S}) \quad \forall K \in \mathcal{T}_h\}. \tag{5}$$

That is the full $C^{\mathrm{div}} - P_{k+1}$ space.

Using Ciarlet's notation, a finite element is usually defined by a triple $(\mathcal{T}, \mathcal{P}_K, \mathcal{L}_K)$.

- A triangulation $\mathcal{T}$ consisting of polyhedral elements K;
- For each K, a space of shape function $\mathcal{P}_K$;
- For each K, a set of local DoFs $\mathcal{L}_K$: a set of functionals on $\mathcal{P}_K$, each associated to a face of T. These must unisolvent, i.e. form a basis for $\mathcal{P}(K)'$

Hu-Zhang's Stress Space

$$\Sigma_{h,k+1}^{HZ} = \{\boldsymbol{\tau} \in H(\Omega; \mathrm{div}, \mathbb{S}) \mid \boldsymbol{\tau}|_K \in P_{k+1}(K; \mathbb{S}), \quad \boldsymbol{\tau}|_a \in C^0, \quad \forall K \in \mathcal{T}_h, a \in \mathcal{N}_h\}. \tag{6}$$

The local DoFs are defined as follows

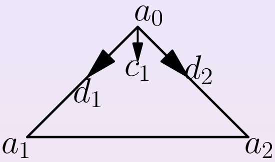
$$\boldsymbol{\sigma}(a) \quad \text{for all vertices } a \text{ of } K. \tag{7a}$$

$$\int_e \boldsymbol{\sigma}\boldsymbol{n} \cdot \boldsymbol{v}\, ds \quad \text{for all } \boldsymbol{v} \in P_{k-1}(e; \mathbb{R}^2), \tag{7b}$$

$$\int_K \boldsymbol{\sigma} : \boldsymbol{\tau}\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{\tau} \in P_{k-1}(K; \mathbb{S}), \tag{7c}$$

Stress Function on a vertex

A symmetric matrix τ on a_0 can be determined by



two double normal values:

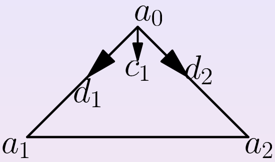
$$d_i := (\boldsymbol{n}_i^T \tau \boldsymbol{n}_i)(a_0), \quad i = 1:2$$

one cross normal values:

$$c_1 := (\boldsymbol{n}_1^T \tau_a \boldsymbol{n}_2)(a_0),$$

Stress Function on a vertex

A symmetric matrix τ on a_0 can be determined by two double normal values:



$$d_i := (\boldsymbol{n}_i^T \boldsymbol{\tau} \boldsymbol{n}_i)(a_0), \quad i = 1:2$$

one cross normal values:

$$c_1 := (\boldsymbol{n}_1^T \boldsymbol{\tau}_a \boldsymbol{n}_2)(a_0),$$

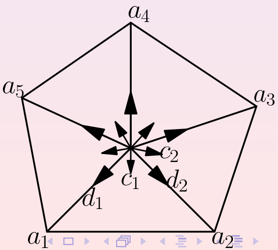
A multi-valued symmetric matrix $\boldsymbol{\tau}_a$ on the $star(a)$ can be determined by following values

the double normal values:

$$d_i := \boldsymbol{n}_i^T \boldsymbol{\tau}_a \boldsymbol{n}_i|_{e_i}, \quad i = 1:n_a,$$

the cross normal values:

$$c_i := \boldsymbol{n}_i^T \boldsymbol{\tau}_a \boldsymbol{n}_{i+1}|_{K_i}, \quad i = 1:n_a.$$



The DoFs on vertices

- One cross normal value:

$$l_{a,c}(\boldsymbol{\sigma}) = \boldsymbol{n}_1^T \boldsymbol{\sigma} \boldsymbol{n}_2|_{K_1}, \tag{8}$$

where K_1 is the element contains e_1, e_2.

The DoFs on vertices

- One cross normal value:

$$l_{a,c}(\boldsymbol{\sigma}) = \boldsymbol{n}_1^T \boldsymbol{\sigma} \boldsymbol{n}_2|_{K_1}, \tag{8}$$

where K_1 is the element contains e_1, e_2.

- the double normal values:

$$l_{a,d}^i(\boldsymbol{\sigma}) = \boldsymbol{n}_i^T \boldsymbol{\sigma} \boldsymbol{n}_i|_{e_i}, \begin{cases} i = 2, 3, \cdots, n_a, & \text{if } a \text{ is an internal non-singular vertex} \\ i = 1, 2, \cdots, n_a, & \text{otherwise.} \end{cases} \tag{9}$$

where n_a is the number of edges meeting at a.

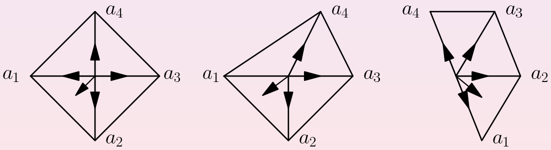


Figure: Singular vertex; non-singular vertex; boundary vertex

The Global DoFs

We then define the global degrees of freedom for the discrete stress space $\Sigma_{h,k+1}$:

$$l_{a,c}(\boldsymbol{\sigma}) = \boldsymbol{n}_1^T \boldsymbol{\sigma} \boldsymbol{n}_2|_{K_1}, \qquad \forall a \in \mathcal{N}_h \tag{10a}$$

$$l_{a,d}^i(\boldsymbol{\sigma}) = \boldsymbol{n}_i^T \boldsymbol{\sigma} \boldsymbol{n}_i|_{e_i} \qquad \forall a \in \mathcal{N}_h \tag{10b}$$

$$l_e^{\boldsymbol{\mu}}(\boldsymbol{\sigma}) = \int_e \boldsymbol{\sigma} \boldsymbol{n} \cdot \mu \, ds, \qquad \forall \boldsymbol{\mu} \in P_{k-1}(e, \mathbb{R}^2), e \in \mathcal{E}_h. \tag{10c}$$

$$l_K^{\tau}(\boldsymbol{\sigma}) = \int_K \boldsymbol{\sigma} : \boldsymbol{\tau} \, d\boldsymbol{x}, \qquad \forall \boldsymbol{\tau} \in P_{k-1}(K; \mathbb{S}), K \in \mathcal{T}_h, \tag{10d}$$

where K_1 and the range of i are defined in (8) and (9).

Unisolvent

Proposition

The set of global DoFs defined above is unisolvent for the space $\Sigma_{h,k+1}$. And the dimension of $\Sigma_{h,k+1}$ is

$$dim(\Sigma_{h,k+1}) = \frac{3}{2}k(k+1)T + 2(k+1)E + E^{\partial} + S, \qquad (11)$$

where T is the number of elements of $\mathcal{T}_h$, E the number of total edges, E^{∂} the number of the boundary edges and S the number of the internal singular vertices.

Outline

1. Mixed Methods for Linear Elasticity
2. A Nodal Basis for the Space $H(\mathrm{div}, \mathbb{S}) \cap \mathcal{P}_{k+1}$
3. A Mixed Finite Element Method
4. Hybridization
5. Numerical Results

Mixed Formulation

We take the discrete displacement space as the full $C^{-1} - \mathcal{P}_k$ space, i.e.

$$\boldsymbol{V}_{h,k} = V_{h,k} \times V_{h,k} \tag{12}$$

where $V_{h,k} = \{v \in L^2(\Omega; \mathbb{R}) \mid v|_K \in P_k(K, \mathbb{R})\}$.

Mixed Formulation

We take the discrete displacement space as the full $C^{-1} - \mathcal{P}_k$ space, i.e.

$$\boldsymbol{V}_{h,k} = V_{h,k} \times V_{h,k} \tag{12}$$

where $V_{h,k} = \{v \in L^2(\Omega; \mathbb{R}) \mid v|_K \in P_k(K, \mathbb{R})\}$.

The mixed finite element approximation of the elastic problem (2) reads: Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \Sigma_{h,k+1} \times \boldsymbol{V}_{h,k}$ such that

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau})_\Omega + (\boldsymbol{u}_h, \mathrm{div}\boldsymbol{\tau})_\Omega = \langle \boldsymbol{g}, \boldsymbol{\tau}\boldsymbol{n}\rangle_{\partial\Omega}, \\ (\mathrm{div}\boldsymbol{\sigma}_h, \boldsymbol{v})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \end{cases} \tag{13}$$

for any $(\boldsymbol{\tau}, \boldsymbol{v}) \in \Sigma_{h,k+1} \times \boldsymbol{V}_{h,k}$.

Stability

According to the theory of mixed method, the stability of the saddle point problem is the corollary of the following two conditions (Brezzi, 1974, 1991)

1. K-ellipticity: There exits a constant $C \geq 0$, independent of the mesh size such that

$$(\mathcal{A}\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq C\|\boldsymbol{\tau}_h\|^2_{\mathrm{div},h}, \quad \forall \boldsymbol{\tau}_h \in Z_h. \tag{14}$$

2. The discrete inf-sup condition: There exits a constant $C \geq 0$, independent of the mesh size such that

$$\inf_{v_h \in V_h} \sup_{\boldsymbol{\tau}_h \in \Sigma_h} \frac{b_h(\boldsymbol{\tau}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{\mathrm{div},h}\|v_h\|_0} \geq C. \tag{15}$$

A discrete elastic complex

$$0 \longrightarrow P_1 \xrightarrow{\hookrightarrow} Q_{h,k+3} \xrightarrow{\text{curl curl}} \Sigma_{h,k+1} \xrightarrow{\text{div}} V_{h,k} \longrightarrow 0$$

Figure: discrete elastic complex

where

$$Q_{h,k+3} = \{q \in H^2(\Omega) \mid q|_K \in P_{k+3}, \forall K \in \mathcal{T}_h\}.$$

A discrete elastic complex

$$0 \longrightarrow P_1 \xrightarrow{\hookrightarrow} Q_{h,k+3} \xrightarrow{\text{curl curl}} \Sigma_{h,k+1} \xrightarrow{\text{div}} V_{h,k} \longrightarrow 0$$

Figure: discrete elastic complex

where

$$Q_{h,k+3} = \{q \in H^2(\Omega) \mid q|_K \in P_{k+3}, \forall K \in \mathcal{T}_h\}.$$

Strang's conjecture: $k+3 \geq 5$(Scott, Morgan, 1975); $k+3=4$(Alfeld, Piper and Schumaker, 1988); $k+3=3$(Scott, Morgan, 1996);

$$\dim(Q_{h,k+3}) \geq \frac{1}{2}(k+4)(k+5)T - (2k+7)E^i + 3V^i + S$$

A discrete elastic complex

$$0 \longrightarrow P_1 \xrightarrow{\hookrightarrow} Q_{h,k+3} \xrightarrow{\text{curl curl}} \Sigma_{h,k+1} \xrightarrow{\text{div}} V_{h,k} \longrightarrow 0$$

Figure: discrete elastic complex

where

$$Q_{h,k+3} = \{q \in H^2(\Omega) \mid q|_K \in P_{k+3}, \forall K \in \mathcal{T}_h\}.$$

Strang's conjecture: $k+3 \geq 5$(Scott, Morgan, 1975); $k+3=4$(Alfeld, Piper and Schumaker, 1988); $k+3=3$(Scott, Morgan, 1996);

$$\dim(Q_{h,k+3}) \geq \frac{1}{2}(k+4)(k+5)T - (2k+7)E^i + 3V^i + S$$

If the equality hold, we have

$$\dim P_1 - \dim Q_{h,k+3} + \dim \Sigma_{h,k+1} - \dim V_{h,k} = 0,$$

which means the div operator is surjective.

Comparison

$$0 \longrightarrow P_1 \xrightarrow{\subset} \tilde{Q}_{h,k+3} \xrightarrow{\mathrm{curl\ curl}} \Sigma^{AW}_{h,k+1} \xrightarrow{\mathrm{div}} V_{h,k-1} \longrightarrow 0$$

$$0 \longrightarrow P_1 \xrightarrow{\subset} \tilde{Q}_{h,k+3} \xrightarrow{\mathrm{curl\ curl}} \Sigma^{HZ}_{h,k+1} \xrightarrow{\mathrm{div}} V_{h,k} \longrightarrow 0$$

$$0 \longrightarrow P_1 \xrightarrow{\subset} Q_{h,k+3} \xrightarrow{\mathrm{curl\ curl}} \Sigma_{h,k+1} \xrightarrow{\mathrm{div}} V_{h,k} \longrightarrow 0$$

Figure: discrete elastic complexes

Comparison

$$0 \longrightarrow P_1 \xrightarrow{\subset} \tilde{Q}_{h,k+3} \xrightarrow{\text{curl curl}} \Sigma^{AW}_{h,k+1} \xrightarrow{\text{div}} V_{h,k-1} \longrightarrow 0$$

$$0 \longrightarrow P_1 \xrightarrow{\subset} \tilde{Q}_{h,k+3} \xrightarrow{\text{curl curl}} \Sigma^{HZ}_{h,k+1} \xrightarrow{\text{div}} V_{h,k} \longrightarrow 0$$

$$0 \longrightarrow P_1 \xrightarrow{\subset} Q_{h,k+3} \xrightarrow{\text{curl curl}} \Sigma_{h,k+1} \xrightarrow{\text{div}} V_{h,k} \longrightarrow 0$$

Figure: discrete elastic complexes

where

$$\tilde{Q}_{h,k+3} = \{q \in H^2(\Omega) \mid q|_K \in P_{k+3}, q|_a \in C^2, \forall K \in \mathcal{T}_h, a \in \mathcal{N}_h\},$$

$$\Sigma^{AW}_{h,k+1} = \{\boldsymbol{\tau} \in H(\Omega; \text{div}, \mathbb{S}) \mid \boldsymbol{\tau}|_K \in P_{k+1}(K; \mathbb{S}), \ \ \text{div}\boldsymbol{\tau}|_K \in P_{k-1}(K; \mathbb{R}^2), \quad \boldsymbol{\tau}|_a \in C^0, \quad \forall K \in \mathcal{T}_h, a \in \mathcal{N}_h\}. \tag{16}$$

Inf-sup Condition I

Lemma

For $k = 0, 1$, suppose $\mathcal{P}_{k+2} - \mathcal{P}_{k+1}$ are stable for the Stokes problem on the grids. For any $\boldsymbol{v}_h \in \boldsymbol{V}_k$, there exists $\boldsymbol{\sigma}_h \in \Sigma_{h,k+1}$ such that

$$\operatorname{div} \boldsymbol{\sigma}_h = \boldsymbol{v}_h \qquad \text{and} \qquad \|\boldsymbol{\sigma}_h\|_{\mathrm{div}} \lesssim \|\boldsymbol{v}_h\|_0. \tag{17}$$

Proof.

- Using the BDM element to find $\boldsymbol{\tau}_h \in H(\mathrm{div}, \Omega; \mathbb{M}) \cap P_{k+1}^{-1}(\mathcal{T}_h; \mathbb{M})$ such that

$$\mathrm{div}\boldsymbol{\tau}_h = \boldsymbol{v}_h \text{ and } \|\boldsymbol{\tau}_h\|_{\mathrm{div}} \lesssim \|\boldsymbol{v}_h\|_0. \tag{18}$$

- Adding a divergence free term to symmetrize $\boldsymbol{\tau}_h$, i.e.

$$\boldsymbol{\sigma}_h = \boldsymbol{\tau}_h + \mathrm{curl}\boldsymbol{u}_h \tag{19}$$

where

$$\mathrm{div}\boldsymbol{u}_h = \tau_{h,12} - \tau_{h,21}. \text{ and } \|\boldsymbol{u}_h\|_1 \lesssim \|\tau_h\|_0. \tag{20}$$

Then we can use the result for the Stokes problem.

□

Inf-sup Condition II

Lemma

For $k \geq 2$, suppose the grids are shape regular. For any $\boldsymbol{v}_h \in \boldsymbol{V}_k$, there exists $\boldsymbol{\sigma}_h \in \Sigma_{h,k+1}$ such that

$$\operatorname{div} \boldsymbol{\sigma}_h = \boldsymbol{v}_h \qquad \text{and} \qquad \|\boldsymbol{\sigma}_h\|_{\operatorname{div}} \lesssim \|\boldsymbol{v}_h\|_0. \tag{21}$$

This is the corollary of the result of Hu-Zhang (2014).

$k = 0, 1$	$k \geq 2$
Special grids	General grids

Table: Constrain on grids

On special grids, the pair $\mathcal{P}_3 - \mathcal{P}_2$ and $\mathcal{P}_2 - \mathcal{P}_1$ is stable for Stokes problem.(Arnold, Qin, 1992)

Stability and Convergence Theorem

Theorem

Suppose the grids satisfy the corresponding condition in Lemma 2 or 3. For any $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$*, the discrete problem* (13) *is uniformly well-posed for* $(\Sigma_{h,k+1}, \|\cdot\|_{\mathrm{div}})$ *and* $(V_{h,k}\|\cdot\|_0)$.

Theorem

Suppose the grids satisfy the corresponding condition in Lemma 2 or 3. Let $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \Sigma \times V$ *be the exact solution of the problem* (2) *and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \Sigma_{h,k+1} \times V_{h,k}$ *the finite element solution of* (13). *Then we have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{div}} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \lesssim h^{k+1}(|\boldsymbol{\sigma}|_{k+2} + |\boldsymbol{u}|_{k+1}) \tag{22}$$

Outline

1 Mixed Methods for Linear Elasticity

2 A Nodal Basis for the Space $H(\mathrm{div}, \mathbb{S}) \cap \mathcal{P}_{k+1}$

3 A Mixed Finite Element Method

4 Hybridization

5 Numerical Results

Hybridization

Hybridization removes the interelement continuity requirements from the space $\Sigma_{h,k+1}$ and places them as an additional equation.

Hybridization

Hybridization removes the interelement continuity requirements from the space $\Sigma_{h,k+1}$ and places them as an additional equation.

Accordingly, we need the space

$$\Sigma_{h,k+1}^{-1} = \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}) \mid \boldsymbol{\tau}|_K \in P_{k+1}(K;\mathbb{S}) \quad \forall K \in \mathcal{T}_h\} \tag{23}$$

without any interelement continuity constraints, as well as a Lagrangian multiplier space

$$\begin{aligned} J_{h,k+1} = \{\boldsymbol{\mu} \in L^2(\mathcal{E}_h,\mathbb{R}^2) \mid \boldsymbol{\mu}|_e = [\boldsymbol{\tau}]|_e \quad \forall \boldsymbol{\tau} \in \Sigma_{h,k+1}^{-1}, e \in \mathcal{E}_h^i \\ \text{and } \boldsymbol{\mu}|_{\mathcal{E}_h^\partial} = 0\}, \end{aligned} \tag{24}$$

which is defined on the edges.

Variational Form

The approximation solution given by the hybridized method is $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k} \times J_{h,k+1}$, satisfying

$$a(\boldsymbol{\sigma}_h, \boldsymbol{u}_h; \boldsymbol{\tau}, \boldsymbol{v}) + b(\boldsymbol{\tau}, \boldsymbol{v}; \boldsymbol{\lambda}_h) = f(\boldsymbol{\tau}, \boldsymbol{v}) \quad (25a)$$

$$b(\boldsymbol{\sigma}_h, \boldsymbol{u}_h; \boldsymbol{\mu}) = 0 \quad (25b)$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\mu}) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k} \times J_{h,k+1}$. The bilinear forms in the system are defined as

$$a(\boldsymbol{\sigma}, \boldsymbol{u}; \boldsymbol{\tau}, \boldsymbol{v}) := (\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega + (\boldsymbol{u}, \mathrm{div}\boldsymbol{\tau})_\Omega + (\mathrm{div}\boldsymbol{\sigma}, \boldsymbol{v})_\Omega \quad (26)$$

$$b(\boldsymbol{\sigma}, \boldsymbol{u}; \boldsymbol{\mu}) := -\langle [\boldsymbol{\sigma}], \boldsymbol{\mu} \rangle_{\mathcal{E}_h^i} \quad (27)$$

$$f(\boldsymbol{\tau}, \boldsymbol{v}) := (\boldsymbol{f}, \boldsymbol{v})_\Omega \quad (28)$$

Equivalence

Proposition

There is a unique solution $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k} \times J_{h,k+1}$ *for the hybridized system* (25)*. Moreover, the first two components of the solution coincide with that of the mixed method* (13)*.*

Proof.

Taking it as a saddle point problem, we can use the standard LBB theory to prove the proposition. □

Characterization of the Multiplier Space

To characterize the multiplier space, we define the following auxiliary spaces

$$M_{h,k+1}^{-1} = \{\boldsymbol{\mu} \in L^2(\mathcal{E}_h, \mathbb{R}^2) \mid \boldsymbol{\mu}|_e = P_{k+1}(e, \mathbb{R}^2) \quad \forall e \in \mathcal{E}_h^i \text{ and } \boldsymbol{\mu}|_{\mathcal{E}_h^\partial} = 0\},$$

$$M_{h,k+1} = \{\boldsymbol{\mu} \in M_{h,k+1}^{-1} \mid \boldsymbol{\mu} \text{ satisfies the following property }\}.$$

Characterization of the Multiplier Space

To characterize the multiplier space, we define the following auxiliary spaces

$$M_{h,k+1}^{-1} = \{\boldsymbol{\mu} \in L^2(\mathcal{E}_h, \mathbb{R}^2) \mid \boldsymbol{\mu}|_e = P_{k+1}(e, \mathbb{R}^2) \quad \forall e \in \mathcal{E}_h^i \text{ and } \boldsymbol{\mu}|_{\mathcal{E}_h^\partial} = 0\},$$

$$M_{h,k+1} = \{\boldsymbol{\mu} \in M_{h,k+1}^{-1} \mid \boldsymbol{\mu} \text{ satisfies the following property }\}.$$

Property: at any internal singular vertex $\boldsymbol{x}_0$ of $\mathcal{T}_h$,

$$\sum_{i=1}^{4} (-1)^i \boldsymbol{\mu}_i \cdot \boldsymbol{n}_{i+1} = 0, \qquad (29)$$

where $\boldsymbol{\mu}_i = \boldsymbol{\mu}|_{e_i}(\boldsymbol{x}_0)$ and $e_i = \overrightarrow{\boldsymbol{x}_0\boldsymbol{x}_i}, i = 1:4$ are the edges meeting at $\boldsymbol{x}_0$, and $\boldsymbol{n}_i$ is the unit normal vector of e_i, as shown in Figure 27.

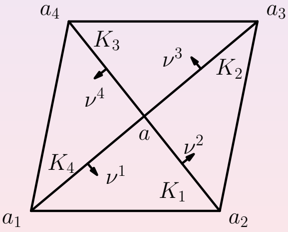


Figure: Singular vertex

Characterization of the Multiplier Space

Proposition

For any triangulation grids $\mathcal{T}_h$, we have

$$J_{h,k+1} = M_{h,k+1} \tag{30}$$

Proof.

- $J_{h,k+1} \subset M_{h,k+1}$.
- $\dim J_{h,k+1} = \dim M_{h,k+1}$.

□

SPD System for $\boldsymbol{\lambda}_h$

Using the above notation, we have the following system

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} (\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \\ \boldsymbol{\lambda}_h \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \tag{31}$$

where A is a block-diagonal matrix.

SPD System for $\boldsymbol{\lambda}_h$

Using the above notation, we have the following system

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} (\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \\ \boldsymbol{\lambda}_h \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \tag{31}$$

where A is a block-diagonal matrix.

Using the Schur complement, we can derive a equation only involved one variable $\boldsymbol{\lambda}_h$, i.e. the multiplier:

$$(-BA^{-1}B^t)\boldsymbol{\lambda}_h = -BA^{-1}F. \tag{32}$$

The original variable can be locally recover by

$$(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) = A^{-1}(F - B^t\boldsymbol{\lambda}_h) \tag{33}$$

Local solver

For any $m \in M_{h,k+1}$, we denote the solution of the following equation system (34) by $(\boldsymbol{\sigma}_m, \boldsymbol{u}_m) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k}$

$$(\mathcal{A}\boldsymbol{\sigma}_m, \boldsymbol{\tau})_K + (\boldsymbol{u}_m, \mathrm{div}\boldsymbol{\tau})_K = \langle m, \boldsymbol{\tau}\boldsymbol{n}\rangle_{\partial K}, \tag{34a}$$

$$-(\mathrm{div}\boldsymbol{\sigma}_m, \boldsymbol{v})_K = 0, \tag{34b}$$

for any $(\boldsymbol{\tau}, \boldsymbol{v}) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k}$, $K \in \mathcal{T}_h$.

Local solver

For any $m \in M_{h,k+1}$, we denote the solution of the following equation system (34) by $(\boldsymbol{\sigma}_m, \boldsymbol{u}_m) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k}$

$$(\mathcal{A}\boldsymbol{\sigma}_m, \boldsymbol{\tau})_K + (\boldsymbol{u}_m, \mathrm{div}\boldsymbol{\tau})_K = \langle m, \boldsymbol{\tau}\boldsymbol{n}\rangle_{\partial K}, \tag{34a}$$

$$-(\mathrm{div}\boldsymbol{\sigma}_m, \boldsymbol{v})_K = 0, \tag{34b}$$

for any $(\boldsymbol{\tau}, \boldsymbol{v}) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k}$, $K \in \mathcal{T}_h$.

For any $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^2)$, we denote the solution of the following equation system (35) by $(\tilde{\boldsymbol{\sigma}}_{\boldsymbol{f}}, \tilde{\boldsymbol{u}}_{\boldsymbol{f}}) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k}$

$$(\mathcal{A}\tilde{\boldsymbol{\sigma}}_{\boldsymbol{f}}, \boldsymbol{\tau})_K + (\tilde{\boldsymbol{u}}_{\boldsymbol{f}}, \mathrm{div}\boldsymbol{\tau})_K = 0, \tag{35a}$$

$$-(\mathrm{div}\tilde{\boldsymbol{\sigma}}_{\boldsymbol{f}}, \boldsymbol{v})_K = -(\boldsymbol{f}, \boldsymbol{v})_K, \tag{35b}$$

for any $(\boldsymbol{\tau}, \boldsymbol{v}) \in \Sigma_{h,k+1}^{-1} \times \boldsymbol{V}_{h,k}$, $K \in \mathcal{T}_h$.

The SPD system for the Multiplier

Theorem

The Lagrangian multiplier $\boldsymbol{\lambda}_h$ satisfies following equation

$$\tilde{a}(\boldsymbol{\lambda}_h, \boldsymbol{\mu}) = \tilde{f}(\boldsymbol{\mu}) \qquad \forall \boldsymbol{\mu} \in J_{h,k+1}, \tag{36}$$

where $\tilde{a}(\boldsymbol{\lambda}_h, \boldsymbol{\mu}) = (\mathcal{A}\boldsymbol{\sigma}_{\boldsymbol{\lambda}}, \boldsymbol{\sigma}_{\boldsymbol{\mu}})_{\mathcal{T}_h}$ *and* $\tilde{f}(\boldsymbol{\mu}) = -(\boldsymbol{f}, \boldsymbol{u}_{\boldsymbol{\mu}})_{\mathcal{T}_h}$. *Moreover, the bilinear form* $a_h(\boldsymbol{\lambda}, \boldsymbol{\mu})$ *in* (36) *is symmetric positive-definite. The solution of the system* (25) *satisfies*

$$\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_{\boldsymbol{\lambda}_h} + \tilde{\boldsymbol{\sigma}}_{\boldsymbol{f}} \qquad \text{and} \qquad \boldsymbol{u}_h = \boldsymbol{u}_{\boldsymbol{\lambda}_h} + \tilde{\boldsymbol{u}}_{\boldsymbol{f}}. \tag{37}$$

Outline

1 Mixed Methods for Linear Elasticity

2 A Nodal Basis for the Space $H(\mathrm{div}, \mathbb{S}) \cap \mathcal{P}_{k+1}$

3 A Mixed Finite Element Method

4 Hybridization

5 Numerical Results

Numerical Results

We consider the pure displacement problem on the unit square $\Omega = [0, 1]^2$ with homogeneous boundary condition.

The compliance tensor in our computation is

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + 2\tau}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right),$$

where the Lamé constants are set to be $\mu = 1/2$ and $\lambda = 1$.

Let the exact solution be

$$u = \begin{pmatrix} \mathrm{e}^{x-y}xy(1-x)(1-y) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix}. \tag{38}$$

Tests for the Lowest Order

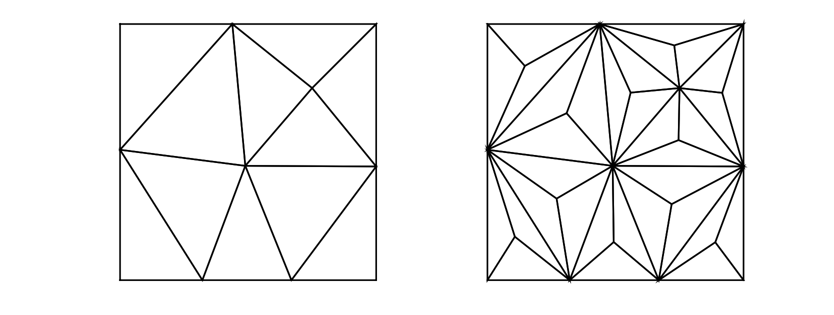


Figure: Unstructured grid and HCT grid with $1/h = 2$

Tests for the Lowest Order

$1/h$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	h^n	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$	h^n	$\|\mathrm{div}\boldsymbol{\sigma}-\mathrm{div}\boldsymbol{\sigma}_h\|_0$	h^n	D[1]	dim $M_{h,1}$
2	1.5654e-1	–	5.0400e-1	–	4.2952e-0	–	363	180
4	9.5309e-2	0.71	2.0147e-1	1.32	2.6589e-0	0.69	1089	2412
8	4.5289e-2	1.07	4.9971e-2	2.01	1.2995e-0	1.03	4554	2412
16	2.2009e-2	1.04	1.2357e-2	2.01	6.3735e-1	1.02	18612	10012
32	1.0976e-2	1.00	3.1761e-3	1.96	3.1827e-1	1.00	74447	40328
64	5.4797e-3	1.00	8.0961e-4	1.97	1.5892e-1	1.00	297264	161592

Table: The errors and the convergence order on HCT grids using hybridized method with $k=0$

$1/h$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	h^n	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$	h^n	$\|\mathrm{div}\boldsymbol{\sigma}-\mathrm{div}\boldsymbol{\sigma}_h\|_0$	h^n	D	dim $M_{h,1}$
2	2.0098e-1	–	6.6940e-1	–	5.4550e-0	–	121	48
4	1.2864e-1	0.64	2.9520e-1	1.18	3.5429e-0	0.62	374	168
8	6.0929e-2	1.07	7.6893e-2	1.94	1.7416e-0	1.02	1518	756
16	2.9574e-2	1.04	2.7646e-2	1.47	8.5519e-1	1.02	6204	3244
32	1.4742e-2	1.00	1.2393e-2	1.15	4.2699e-1	1.00	24816	13256
64	7.3592e-3	1.00	6.0029e-3	1.04	2.1325e-1	1.00	99088	53496

Table: The errors and the convergence order on unstructured grids using hybrided method with $k=0$

[1] $D = \dim \Sigma_{h,1}^{-1} + \dim V_{h,0}$

Tests for the Higher Order

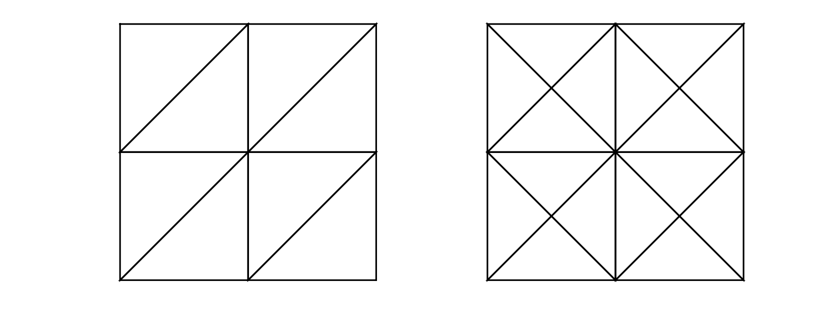


Figure: Uniform grid and criss-cross grid with $1/h = 2$

Tests for the Higher Order

$1/h$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	h^n	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$	h^n	$\|\mathrm{div}\boldsymbol{\sigma}-\mathrm{div}\boldsymbol{\sigma}_h\|_0$	h^n	D^2	$\dim M_{h,1}$
2	1.6541e-2	–	3.0057e-2	–	4.7388e-1	–	336	64
4	2.1758e-3	2.92	2.0260e-3	3.73	6.2558e-2	2.92	1344	320
8	2.7561e-4	2.98	1.5145e-4	3.89	7.9274e-3	2.98	5376	1408
16	3.4569e-5	2.99	9.7454e-6	3.95	9.9431e-4	2.99	21504	5888
32	4.3248e-6	2.99	6.1737e-7	3.98	1.2439e-4	3.00	86016	24064
64	5.4072e-7	3.00	3.8838e-8	3.99	1.5552e-5	3.00	344064	97280

Table: The errors and the convergence order on uniform grids using hybridized method with $k = 2$.

$1/h$	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	h^n	$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0$	h^n	$\|\mathrm{div}\boldsymbol{\sigma}-\mathrm{div}\boldsymbol{\sigma}_h\|_0$	h^n	D	$\dim M_{h,1}$
2	4.5314e-3	–	4.7780e-3	–	1.3437e-1	–	672	160
4	5.7633e-4	2.97	3.1371e-4	3.92	1.7027e-2	2.98	2688	704
8	7.2355e-5	2.99	2.0057e-5	3.96	2.1361e-3	2.99	10752	2944
16	9.0541e-6	2.99	1.2672e-6	3.98	2.6726e-4	2.99	43008	12032
32	1.1320e-6	3.00	7.9629e-8	3.99	3.3416e-4	3.00	172032	48640
64	1.4151e-7	3.00	4.9899e-9	4.00	4.1772e-5	3.00	688128	195584

Table: The errors and the convergence order on criss-cross grids using hybridized method with $k = 2$.

[2] $D = \dim \Sigma_{h,1}^{-1} + \dim V_{h,0}$

Conclusion

- We propose a nodal basis for the stress space $H(\mathrm{div}, \mathbb{S}) \cap \mathcal{P}_k$.
- We prove optimal error estimate of our mixed method for both displacement and stress.
- Our method can be efficiently implemented by hybridization.

Symmetric, Conforming and Hybridizable Elements!

Thank you!